On the Construction of Zero Energy States in Supersymmetric Matrix Models II

Jens Hoppe*
Institute for Theoretical Physics
ETH Hönggerberg
CH-8093 Zürich

September/October 1997

Abstract

For the $SU(N)$ invariant supersymmetric matrix model related to membranes in 11 space-time dimensions, the general (bosonic) solution to the equations $Q^{\dagger}_{\beta}\Psi = 0$ ($Q_{\beta}\Psi = 0$) is determined.

*Heisenberg Fellow

Continuing [1], I present the explicit form of all $SU(N)$-invariant wave functions

$$\Psi \;=\; \psi \;+\; \frac{1}{2}\,\lambda_a\lambda_b\,\psi_{ab} \;+\; \cdots \;+\; \frac{\lambda_{a_1}\cdots\lambda_{a_\Lambda}}{\Lambda!}\,\psi_{a_1\cdots a_\Lambda} \qquad , \tag{1}$$

$\Lambda \;=\; 8(N^2-1)$, $\{\lambda_a,\lambda_b\} = 0 = \{\partial_{\lambda_a},\partial_{\lambda_b}\}$, $\{\lambda_a,\partial_{\lambda_b}\} = \delta_{ab}$, satisfying $Q^\dagger_\beta\Psi = 0$ ($Q_\beta\Psi = 0$), $\beta = 1,\ldots,8$ arbitrary (but fixed), when

$$Q_\beta \;=\; M^{(\beta)}_a\,\lambda_a \;+\; D^{(\beta)}_a\,\partial_{\lambda_a} \;\;,\quad Q^\dagger_\beta \;=\; M^{(\beta)\,\dagger}_a\,\partial_{\lambda_a} \;+\; D^{(\beta)\,\dagger}_a\,\lambda_a \tag{2}$$

with

$$M^{(\beta)}_{\alpha A} \;=\; \delta_{\alpha\beta}\,iq_A \;+\; i\Gamma^j_{\alpha\beta}\,\frac{\partial}{\partial x_{jA}} \;-\; \frac{1}{2}\,f_{ABC}\,x_{jB}\,x_{kC}\,\Gamma^{jk}_{\alpha\beta} \tag{3}$$

$$D^{(\beta)}_{\alpha A} \;=\; \delta_{\alpha\beta}\,2\partial_A \;-\; if_{ABC}\,x_{jB}\,\overline{z_C}\Gamma^j_{\alpha\beta} \qquad , \tag{4}$$

$\alpha,\beta = 1,\ldots,8$, $j,k,l = 1,\ldots,7$, $A,B,C = 1,\ldots,N^2-1$, $\{\Gamma^j,\Gamma^k\} = 2\delta^{jk}\mathbb{1}$, $\overline{\Gamma^j} = -\Gamma^j$.

In order to resolve the non-commutativity of M with $M^\dagger$, consider

$$\tilde{Q}^\dagger_\beta \;:=\; F^{-1}_\beta\,Q^\dagger_\beta\,F_\beta \tag{5}$$

with

$$F_\beta \;=\; \exp\left(\frac{1}{6}\,f_{ABC}\,x_{jA}\,x_{kB}\,x_{lC}\,u_{jkl}(\beta)\right) \qquad . \tag{6}$$

While (6) commutes with $D^\dagger$, the unwanted 3rd term in

$$M^{(\beta)\,\dagger}_{\alpha A} \;=\; -i\delta_{\alpha\beta}\,q_A \;-\; i\Gamma^j_{\alpha\beta}\,\frac{\partial}{\partial x_{jA}} \;-\; \frac{1}{2}\,f_{ABC}\,x_{jB}\,x_{kC}\,\Gamma^{jk}_{\alpha\beta} \tag{7}$$

will be removed, provided

$$\Gamma^j_{\alpha\beta}\,u_{jkl}(\beta) \;=\; i\Gamma^{kl}_{\alpha\beta} \tag{8}$$

for all α, k, l (and arbitrary, but fixed β).

In the representation

$$i\hat{\Gamma}^j_{k8} \;:=\; \delta_{jk} \;\;,\quad i\hat{\Gamma}^j_{kl} \;:=\; -c_{jkl} \qquad , \tag{9}$$

where the c_{jkl} (totally antisymmetric) are octonionic structure constants satisfying

$$c_{jkl}\,c_{mnl} \;=\; \delta^j_m\,\delta^k_n \;-\; \delta^j_n\,\delta^k_m \;-\; \frac{1}{6}\,\epsilon_{jkmnrst}\,c_{rst} \tag{10}$$

(cp. [2], where this representation was already used in a truncation of the 11-dimensional model to $N = 1$ sypersymmetry, with a non-normalizable zero-energy state of the form (6)) the solution of (8) turns out to be

$$\hat{u}_{jkl}(\beta) \;=\; c_{jkl}\,\begin{cases} +1 & \text{if } \beta = j,k,l \text{ or } 8 \\ -1 & \text{if } \beta \neq j,k,l \text{ and } 8 \end{cases} \tag{11}$$

(which is easy to check, using $\hat{\Gamma}^{jk}_{l8} = -c_{jkl}$ and $\hat{\Gamma}^{kl}_{mn} = \delta^k_m \delta^l_n - \delta^k_n \delta^l_m + \frac{1}{6}\epsilon_{klmnrst}c_{rst}$; the representation invariant form of (11) is $i(\Gamma^{jkl})_{\beta\beta}$).

In order to solve the equation $Q^\dagger_\beta \Psi = 0$ ($Q_\beta \Psi = 0$), consider $\tilde{Q}^\dagger_\beta (F^{-1}_\beta \Psi) = 0$, i.e.

$$\left(\left(-i\delta_{\alpha\beta}\, q_A \;-\; i\Gamma^j_{\alpha\beta}\, \frac{\partial}{\partial x_{jA}}\right) \frac{\partial}{\partial \lambda_{\alpha A}} \;+\; D^{(\beta)\,\dagger}_{\alpha A}\, \lambda_{\alpha A}\right) \tilde{\psi} \;=\; 0 \quad , \tag{12}$$

or, in components,

$$(2k-1)\, D^{(\beta)\,\dagger}_{[a_1}\, \tilde{\psi}_{a_2 \cdots a_{2k-1}]} \;=\; N^{(\beta)\,\dagger}_{a_{2k}}\, \tilde{\psi}_{a_1 \cdots a_{2k}} \qquad , \tag{13}$$

$k = 1, 2, \ldots K := 4(N^2 - 1)$ with

$$N^{(\beta)\,\dagger}_{\alpha A} \;:=\; -i\delta_{\alpha\beta}\, q_A \;-\; i\Gamma^j_{\alpha\beta}\, \frac{\partial}{\partial x_{jA}} \;=\; F^{-1}_\beta\, M^{(\beta)\,\dagger}_{\alpha A}\, F_\beta \;=\; -N^{(\beta)}_{\alpha A} \quad . \tag{14}$$

Using

$$J_E\, \tilde{\psi} \;:=\; -i f_{EAA'} \left(x_{jA}\, \frac{\partial}{\partial x_{jA'}} \;+\; z_A\, \partial_{A'} \;+\; \overline{z_A}\, \overline{\partial_{A'}} \;+\; \lambda_{\alpha A}\, \partial_{\lambda_{\alpha A'}}\right) \tilde{\psi} \;=\; 0 \tag{15}$$

one can show that the <u>general</u> solution of (12)/(13) is

$$\tilde{\psi}_{a_1 \cdots a_{2k}} \;=\; -(2k)(2k-1)\, N^{(\beta)}_{[a_1}\, (N^+ N)^{-1}\, D^{(\beta)\,\dagger}_{a_2} \tilde{\psi}_{a_3 \cdots a_{2k}]} \;+\; \tilde{\psi}^{(h)}_{a_1 \cdots a_{2k}} \quad , \tag{16}$$

hence the general solution of $Q^\dagger_\beta \Psi = 0$

$$\psi_{a_1 \cdots a_{2k}} \;=\; -(2k)(2k-1)\, F_\beta\, N_{[a_1}\, (N^\dagger N)^{-1}\, D^\dagger_{a_2}\, F^{-1}_\beta\, \psi_{a_3 \cdots a_{2k}]} \;+\; \psi^{(h)}_{a_1 \cdots a_{2k}} \quad , \tag{17}$$

where $N^{(\beta)\,\dagger}_{a_{2k}} (F^{-1}_\beta\, \Psi^{(h)}_{a_1 \cdots a_{2k}}) \equiv 0$, i.e. $M^{(\beta)\,\dagger}_{a_{2k}} \psi^{(h)}_{a_1 \cdots a_{2k}} \equiv 0$. Analogously, the general solution of $Q_\beta \Psi = 0$, i.e.

$$(2k-1)\, M^{(\beta)}_{[a_1}\, \psi_{a_2 \cdots a_{2k-1}]} \;=\; D_{a_{2k}}\, \psi_{a_1 \cdots a_{2k}} \quad , \tag{18}$$

is given by

$$\psi_{a_1 \cdots a_{2k-2}} \;=\; F^{-1}_\beta\, (N^\dagger\, N)^{-1}\, N^{(\beta)\,\dagger}_a\, D^{(\beta)}_b\, F_\beta\, \psi_{a_1 \cdots a_{2k-2} ab} \;+\; \psi^{[h]}_{a_1 \cdots a_{2k-2}} \tag{19}$$

with $M^{(\beta)}_{[a}\, \psi^{[h]}_{a_1 \cdots a_{2k-2}]} \equiv 0$.

Perhaps it is useful to present one of the proofs (e.g. that (16) satisfies (13)) explicitly:

$$\begin{aligned}
& N^{(\beta)\,\dagger}_{a_{2k}}\, \tilde{\psi}_{a_1 \cdots a_{2k}} \;-\; (2k-1)\, D^{(\beta)\,\dagger}_{[a_1}\, \tilde{\psi}_{a_2 \cdots a_{2k-1}]} \qquad\qquad (20)\\
&= \; -(2k-1)\, N^{(\beta)\,\dagger}_{a_{2k}}\, N^{(\beta)}_{[a_{2k-1}}\, (N^\dagger\, N)^{-1}\, D^{(\beta)\,\dagger}_{(a_{2k})}\, \tilde{\psi}_{a_1 \cdots a_{2k-2}]} \\
&\qquad -(2k-1)(2k-2)\, (N^\dagger\, N)^{-1}\, N^{(\beta)\,\dagger}_{a_{2k}}\, N^{(\beta)}_{[a_1}\, D^\dagger_{a_2}\, \tilde{\psi}_{a_3 \cdots]a_{2k}} \\
&= \; -(2k-1)\, (N^\dagger\, N)^{-1} N^{(\beta)}_{[a_1} \left\{ \vec{N}^\dagger\, \vec{D}^\dagger\, \tilde{\psi}_{a_2 \cdots a_{2k-1}]} \;+\; (2k-2)\, [N^{(\beta)\,\dagger}_{(a_{2k})}, D^{(\beta)\,\dagger}_{a_2}]\, \tilde{\psi}_{a_3 \cdots]a_{2k}} \right. \\
&\qquad \left. +\, (2k-2)\, D^{(\beta)\,\dagger}_{a_2}\, N^{(\beta)\,\dagger}_{(a_{2k})}\, \tilde{\psi}_{a_3 \cdots]a_{2k}} \right\} \;=\; 0 \quad ,
\end{aligned}$$

as the first two terms inside the bracket combine to give $(-iz_E J_E \tilde{\psi})^{(2k-2)}$ (which is zero) and the last term vanishes by induction hypothesis (i.e. $\tilde{\psi}^{(2k-2)}$ satisfying $(13)_{k\to k-1}$).

Note added: Instead of trying to give an analytical meaning to $N^{(\beta)}_{\alpha A}(N^\dagger N)^{-1}$, one may simply use

$$I^{(\beta)}_{\alpha A} := \delta_{\alpha\beta}\,\frac{iq_A}{q^2} \qquad ,$$

as $N^\dagger_a I_a = \mathbb{1}$ and $[I_a, N^\dagger_b] = 0$ (as well as $[I_a, F_\beta] = 0$, making unnecessary the detour via F).

Acknowledgement: I am very grateful to F. Finster, J. Goldstone, H. Nicolai, S.-T. Yau and E. Zaslow for helpful discussions and to the Mathematics Department of Harvard University for its hospitality.

References

[1] J. Hoppe; hep-th/9709132

[2] B. de Wit, J. Hoppe, H. Nicolai; Nucl. Phys. **B 305** (1988) 545